WEAK RICCI CURVATURE BOUNDS FOR RICCI SHRINKERS

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Abstract. We show that for a complete Ricci shrinker there exists a sequence of points tending to infinity whose norms of the Ricci tensor grow at most linearly.

All objects are C^∞. Let $(\mathcal{M}^n, g)$ be a Riemannian manifold and $\phi, f : \mathcal{M} \to \mathbb{R}$. For $\gamma : [0, \bar{s}] \to \mathcal{M}$, $\bar{s} > 0$, define $S = \gamma'$ and $\mathcal{J}(\gamma) = \int_0^{\bar{s}} \left(|S(s)|^2 + 2\phi(\gamma(s)) \right) ds$. A critical point γ of $\mathcal{J}$ on paths with fixed endpoints, called a ϕ-geodesic, satisfies $\nabla_S S = \nabla\phi$ and $|S|^2 - 2\phi = C$. Let $\mathrm{Rc}_f = \mathrm{Rc} + \nabla\nabla f$. For a minimal ϕ-geodesic,

$$(1) \qquad -\int_0^{\bar{s}} \zeta^2 \Delta_f \phi \, ds + \int_0^{\bar{s}} \zeta^2 \, \mathrm{Rc}_f(S, S) \, ds \leq \int_0^{\bar{s}} \left(n(\zeta')^2 - 2\zeta\zeta' \langle \nabla f, S \rangle \right) ds,$$

where $\Delta_f = \Delta - \nabla f \cdot \nabla$ and $\zeta : [0, \bar{s}] \to \mathbb{R}$ is piecewise C^∞, vanishing at 0 and $\bar{s}$.

Let (g, f) be a complete shrinker and satisfy $\mathrm{Rc}_f = \frac{1}{2} g$ and $f - |\nabla f|^2 = R > 0$. Let $c > 0$ and $2\phi = c\frac{R}{f}$. From $\Delta_f R = -2|\mathrm{Rc}|^2 + R$ and $\Delta_f f = \frac{n}{2} - f$ we compute

$$\Delta_f \frac{R}{f} = \frac{R}{f^2}(2f - \frac{n}{2}) - 2\frac{|\mathrm{Rc}|^2}{f} - 4\frac{\mathrm{Rc}(\nabla f, \nabla f)}{f^2} + 2\frac{R|\nabla f|^2}{f^3} \leq -\frac{|\mathrm{Rc}|^2}{f} + 4\frac{(1+\sqrt{n})^2}{f}.$$

If $\zeta(s) = s$ for $s \in [0,1]$, $\zeta(s) = 1$ for $s \in [1, \bar{s} - 1]$, $\zeta(s) = \bar{s} - s$ for $s \in [\bar{s} - 1, \bar{s}]$, then

$$\frac{c}{2} \int_0^{\bar{s}} \zeta^2 \left(\frac{|\mathrm{Rc}|^2}{f} - 4\frac{(1+\sqrt{n})^2}{f} \right) ds + \frac{1}{2} \int_0^{\bar{s}} \zeta^2 |S|^2 \, ds \leq 2n - \int_0^{\bar{s}} 2\zeta\zeta' \langle \nabla f, S \rangle \, ds.$$

Let $\gamma(0) = x$, $\gamma(\bar{s}) = y$, and $\bar{s} = d(x, y)$. Then $1 - c \leq C \leq 1 + c$; the lower by $\frac{R}{f} \leq 1$ and the upper since for a minimal geodesic $\bar{\gamma}(s)$, $s \in [0, \bar{s}]$, from x and y,

$$C\bar{s} \leq \int_0^{\bar{s}} \left(|\gamma'(s)|^2 + c\frac{R(\gamma(s))}{f(\gamma(s))} \right) ds \leq \int_0^{\bar{s}} \left(|\bar{\gamma}'(s)|^2 + c\frac{R(\bar{\gamma}(s))}{f(\bar{\gamma}(s))} \right) ds \leq (1 + c)\bar{s}.$$

Let $f(O) = \min_{\mathcal{M}} f \leq \frac{n}{2}$ and $r = d(\cdot, O)$. Then $|\nabla f|(z) \leq \sqrt{f(z)} \leq \sqrt{\frac{n}{2}} + r(z)$. Since $|S| \leq \sqrt{C + c}$ and $r(\gamma(s)) \leq \min\{r(x) + s\sqrt{C+c}, r(y) + (\bar{s} - s)\sqrt{C+c}\}$,

$$\begin{aligned} -\int_0^{\bar{s}} \zeta\zeta' \langle \nabla f, S \rangle \, ds &\leq \int_0^1 s\sqrt{f(\gamma(s))}\,|S(s)| \, ds + \int_{\bar{s}-1}^{\bar{s}} (\bar{s} - s)\sqrt{f(\gamma(s))}\,|S(s)| \, ds \\ &\leq \tfrac{1}{2}\sqrt{C+c}\left(\sqrt{2n} + r(x) + r(y) + 2\sqrt{C+c} \right). \end{aligned}$$

Let $A = \sqrt{C + c}$. Since $f(\gamma(s)) \geq f(O)$ and $\bar{s} = d(x, y)$, we have

$$\int_0^{\bar{s}} \frac{\zeta^2 |\mathrm{Rc}|^2}{f} ds \leq \frac{4(1+\sqrt{n})^2 d(x, y)}{f(O)} + \frac{4(\sqrt{n} + A)^2}{c} + \frac{2A(r(x) + r(y))}{c}.$$

Take $x = O$ and $\bar{s} = r(y) \geq 2\sqrt{\frac{n}{2}}$. Then $d(\gamma(s), y) \leq \frac{r(y)}{2}$ for $s \in [\frac{2A-1}{2A}\bar{s}, \bar{s}]$ and

$$\frac{(\frac{r(y)}{2A} - 1) \min\limits_{s \in [(1 - \frac{1}{2A})\bar{s}, \bar{s}]} |\mathrm{Rc}|^2(\gamma(s))}{(\sqrt{\frac{n}{2}} + \frac{3r(y)}{2})^2} \leq \int_{(1-\frac{1}{2A})\bar{s}}^{\bar{s}-1} \frac{|\mathrm{Rc}|^2(\gamma(s))}{f(\gamma(s))} ds \leq \mathrm{Const}\,(r(y) + 1).$$

Thus there exists $C < \infty$ such that for any $y \in \mathcal{M}$ with $r(y) \geq \max\{\sqrt{2n}, 3A\}$, there exists a point $z \in \mathcal{M}$ with $d(z, y) \leq \frac{r(y)}{2}$ and $|\operatorname{Rc}|(z) \leq C(r(y) + 1)$.

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